

# Cryocoolers (CryoCoolers.tex)

A.T.A.M. de Waele  
Eindhoven University of Technology

September 4, 2009

## Abstract

This document describes the main features of cryocoolers in general and pulse-tube refrigerator in particular.

## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>Pulse-tube refrigerators</b>	<b>2</b>
2.1	Components . . . . .	2
2.2	Cooling principle . . . . .	3
2.3	An idealized cycle . . . . .	4
2.4	Ideal regenerators . . . . .	5
2.5	Thermodynamical approach . . . . .	5
<b>3</b>	<b>Joule-Thomson coolers</b>	<b>8</b>
<b>4</b>	<b>Stirling Coolers</b>	<b>12</b>
4.1	Fundamental cycle . . . . .	12
4.2	Displacer-type . . . . .	14
4.3	Split pair type . . . . .	15
<b>5</b>	<b>GM-coolers</b>	<b>16</b>
<b>6</b>	<b>Appendix: Ideal-gas relations</b>	<b>18</b>

# 1 Introduction

Cryocoolers are machines to reach cryogenic temperatures. Usually they are of table-top size and used to cool a particular application. So, usually they are integrated in a larger system in which very low temperatures play an essential role. In this document the basic operation of four types of cryocoolers is described. In particular: pulse-tube refrigerators, Joule-Thomson coolers, Stirling coolers, and Gifford-McMahon (GM) coolers.

## 2 Pulse-tube refrigerators

### 2.1 Components

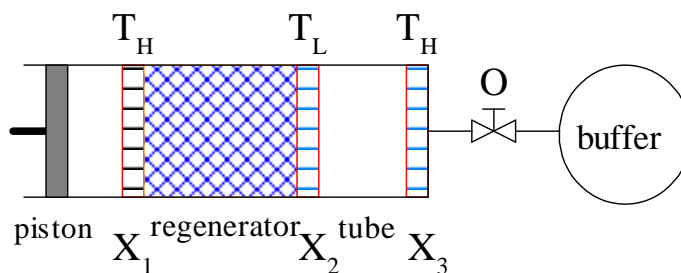


Figure 1: Schematic diagram of a Stirling type single-orifice PTR. From left to right the system consists of a compressor with moving piston (*piston*), the after cooler ( $X_1$ ), a regenerator, a low-temperature heat exchanger ( $X_2$ ), a tube (*tube*), a second room-temperature heat exchanger ( $X_3$ ), an orifice ( $O$ ), and a buffer. The cooling power is generated at the low temperature  $T_L$ . Room temperature is  $T_H$ .

This section gives a description of the basic operation of a Pulse-Tube Refrigerator (PTR) in the steady state. A so-called Stirling-type single-orifice PTR is represented schematically in Fig.1. From left to right it consists of:

1. An externally driven *piston* which moves back and forth.
2. A *heat exchanger*  $X_1$  (after cooler) where heat is released at room temperature ( $T_H$ ) to cooling water or to the surroundings.
3. A *regenerator* which consists of a porous material with a large heat capacity.
4. A *heat exchanger*  $X_2$  at low temperature ( $T_L$ ) where heat is taken up from the application.
5. A *tube*, often called the pulse tube.
6. A *heat exchanger*  $X_3$  to room temperature ( $T_H$ ).
7. A *flow resistance*  $O$  (orifice).
8. A *buffer volume*, in which the pressure is practically constant.

The system is filled with helium at an average pressure of typically 20 bar. The part in between the heat exchangers  $X_1$  and  $X_3$  is below room temperature. It is contained in a vacuum chamber for thermal isolation.

## 2.2 Cooling principle

The piston moves the gas back and forth and generates a varying pressure in the system. The operating frequency typically is 1 to 50 Hz. In this Section all flow resistances are neglected except from the orifice. The pressure varies smoothly. Acoustic effects, such as travelling pressure waves, or fast pressure changes (pulses), are absent. The operation of PTR's has nothing to do with "pulses".<sup>1</sup>

In the regenerator and in heat exchangers the gas is in very good thermal contact with its surroundings while in the tube the gas is thermally isolated. Gas elements inside the tube are compressed or expanded adiabatically and reversibly, so their entropy is constant. Using the general expression for the molar entropy  $S_m$  of the gas

$$TdS_m = C_p dT - T\alpha_V V_m dp, \quad (1)$$

with  $T$  the temperature,  $C_p$  the molar heat capacity at constant pressure,  $\alpha_V$  the volumetric thermal expansion coefficient given by

$$\alpha_V = \frac{1}{V_m} \left( \frac{\partial V_m}{\partial T} \right)_p, \quad (2)$$

$V_m$  the molar volume, and  $p$  the pressure. From Eq.(1), with  $\delta S_m = 0$ , we see that the temperature variation  $\delta T$  is related to a pressure variation  $\delta p$  according to

$$\delta T = \frac{T\alpha_V V_m}{C_p} \delta p \quad (S_m \text{ constant}). \quad (3)$$

Usually (not always!)  $\alpha_V > 0$ . This well-known property means that compression leads to heating and expansion to cooling. This fact is the basis for the operation of many types of coolers.

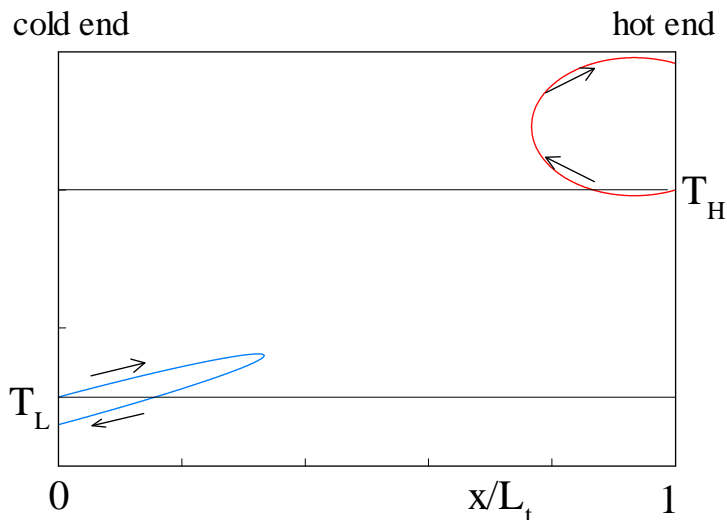


Figure 2: *Left side: a gas element enters the tube at temperature  $T_L$  and leaves it at a lower temperature hence producing cooling. Right side: a gas element enters the tube at temperature  $T_H$  and leaves it at a higher temperature hence producing heating.*

Schematic drawings of the temperature-position curves of two gas elements (one at the cold end and one at the hot end) is given in Fig.2.

<sup>1</sup>The name "pulse-tube refrigerator" is completely wrong, but unfortunately it is so well-established that we just have to live with that.

At the hot end gas flows from the buffer via the orifice into the tube with a temperature  $T_H$  if the pressure  $p_t$  is below the pressure in the buffer  $p_B$  ( $p_t < p_B$ ). If  $p_t = p_B$  the gas at the hot end stops moving. If  $p_t > p_B$  the gas moves to the right through the heat exchanger  $X_3$ . So gas elements enters the tube if  $p_t < p_B$  and leaves the tube if  $p_t > p_B$ . So the final pressure is larger than the initial pressure. Consequently the gas leaves the tube with a temperature higher than the initial temperature  $T_H$  (Eq.(3)). Heat is released via the heat exchanger  $X_3$  to the surroundings and the gas flows to the orifice at ambient temperature.

At the cold end of the tube the gas leaves the cold heat exchanger  $X_2$  with temperature  $T_L$  and enters the tube when the pressure is high. It returns to  $X_2$  when the pressure is low and the temperature is below  $T_L$ . Hence producing cooling.

### 2.3 An idealized cycle

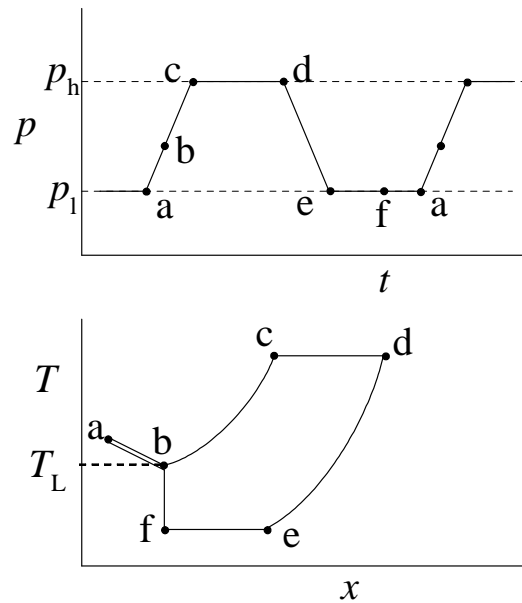


Figure 3: *Top: pressure in the tube as function of time. Bottom: temperature of a gas element as a function of its position in the tube.*

An alternative way to understand the cooling is represented in Fig.3. It illustrates the cooling process at the cold end in a somewhat idealized cycle. The pressure in the tube is assumed to vary in four steps:

1. from a via b to c. The piston moves to the right with the orifice is closed. The pressure rises.
2. c to d. The orifice is opened so that gas flows from the tube to the buffer. At the same time the piston moves to the right in such a way that the pressure in the tube remains constant.
3. d to e. The piston moves to the left with the orifice is closed. The pressure drops.
4. e via f to a. The orifice is opened so that gas flows from the buffer into the tube. At the same time the piston moves to the left so that the pressure in the tube remains constant.

Now we follow a gas element that passes  $X_2$  during the cycle. At the start of the cycle (**point (a)**) it is inside the regenerator.

**a to b:** When the pressure rises the gas element moves to the right but its temperature remains equal to the local temperature due to the good heat contact with the regenerator material.

**At point (b)** our gas element leaves the regenerator, passes  $X_2$ , and enters the tube with the temperature  $T_L$ . The pressure is  $p_b$ .

**b to c:** Now the gas element is inside the tube, so it is thermally isolated. Here its temperature rises together with the pressure while it moves to the right.

**c to d:** The gas element moves to the right. The pressure is constant so the temperature of our gas element is constant.

**d to e:** When the pressure drops the gas element moves to the left. As it is thermally isolated its temperature drops to a value below  $T_L$  since  $p_e < p_b$ .

**e to f:** The gas element moves to the left. The pressure is constant so the temperature is constant.

**At point (f)** the gas element enters the heat exchanger  $X_2$ . In passing  $X_2$  the gas extracts heat (produces cooling) from  $X_2$ . The gas element warms up to the temperature  $T_L$ .

**f to a:** The gas element is inside the regenerator and moves with the local temperature back to its original position.

## 2.4 Ideal regenerators

The thermodynamic and hydrodynamic properties of regenerators are extremely complicated. In many cases it is necessary to make simplifying assumptions. The degree of idealization may differ from case to case. In its most extreme form in an ideal regenerator:

1. the heat capacity of the matrix is much larger than of the gas;
2. the heat contact between the gas and the matrix is perfect;
3. the gas in the regenerator is an ideal gas;
4. the flow resistance of the matrix is zero;
5. the axial thermal conductivity is zero;
6. sometimes it is also assumed that the void volume of the matrix is zero.

Depending on the situation one or more assumptions may be dropped. Usually it is replaced by an other assumption with a less rigorous nature. If conditions 1 and 2 are satisfied then the gas temperature at a certain point in the regenerator is constant. If condition 3 is satisfied as well then the average enthalpy flow in the regenerator is zero as will be shown in the next Section. If conditions 2, 4, and 5 are satisfied there are no irreversible processes in the regenerator.

## 2.5 Thermodynamical approach

In this Section we will analyze the PTR in a very elegant way purely based on the first and second law of thermodynamics. We split up the pulse tube in subsystems going from right to left (Fig.4).

In the ideal case entropy is produced only in the orifice. In all the other subsystems  $\dot{S}_i = 0$ . The heat flows  $\bar{Q}$  are nonzero only in the heat exchangers. Flows to the right are counted positive. We assume that the working fluid is an ideal gas. In that case the enthalpy is only a function of temperature. In fact

$$H_m = C_p T. \quad (4)$$

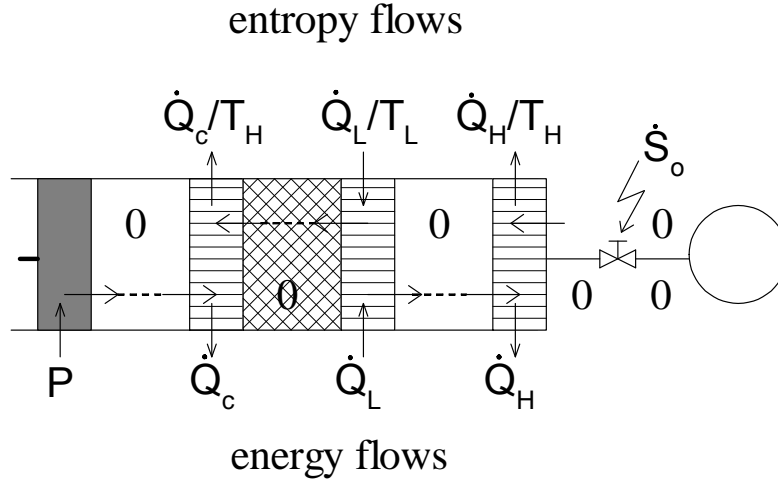


Figure 4: Schematic diagram of a single-orifice PTR. In the upper half of the figure the entropy flows are indicated. The lower half gives the energy flows. If a particular flow is zero in a certain region this is indicated by '0'. It is assumed that the PTR is ideal so that dissipation only takes place in the orifice.

The average enthalpy flow is given by

$$\overline{H}^* = \frac{1}{t_c} \int_0^{t_c} \dot{n} H_m(T) dt \quad (5)$$

where  $t_c$  is the cycle time. In the steady state the average molar flow is zero everywhere in the PTR

$$\overline{\dot{n}}^* = \frac{1}{t_c} \int_0^{t_c} \dot{n} dt = 0. \quad (6)$$

First we consider a very simple system as represented by the dotted rectangle in the left of Fig.5 containing only the orifice. At the right side of the orifice the pressure  $p = p_B$  and the temperature  $T = T_H$  are constant. As a result  $S_m$  and  $H_m$  are constant as well. Using Eq.(6) it follows that the average enthalpy flow in the tube connecting the orifice with the buffer is zero. In the tube connecting the orifice and the heat exchanger  $X_3$  the temperature is constant as well so also here the enthalpy flow is zero. As a result the heat, exchanged between the surroundings and the orifice, is zero.

In the orifice entropy is produced. The average entropy flow in the tube right of the orifice is zero because  $S_m$  is constant and the average molar flow is zero. However, the the average entropy flow in the tube left of the orifice is nonzero due to the pressure dependence of  $S_m$ . Eq.(33) shows that  $S_m$  is low if  $p_t$  is high and  $S_m$  is high if  $p_t$  is low. The entropy flow in the tube left of the orifice is equal to the entropy produced in the orifice. In other words: the entropy, which is produced in the orifice, flows towards the heat exchanger  $X_3$ .

Now we consider the system containing the pulse tube and its two adjacent heat exchangers  $X_2$  and  $X_3$  (Fig.6). The average enthalpy flow in the pulse tube is given by

$$\overline{H_t}^* = C_p \overline{\dot{n} T}^*. \quad (7)$$

The gas moves to the right ( $\dot{n}^* > 0$ ) with a high temperature and to the left ( $\dot{n}^* < 0$ ) with a low temperature. As a result the net enthalpy transport  $\overline{H_t}^*$  is nonzero in the pulse tube. If the regenerator is ideal the

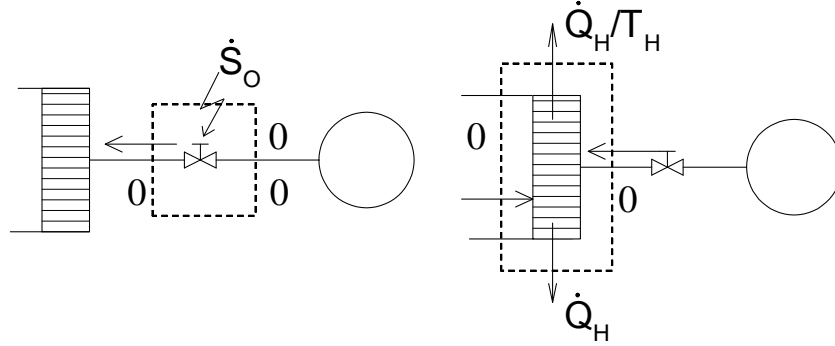


Figure 5: *Thermodynamic systems. In the left the system contains the orifice. The right contains heat the exchanger  $X_3$ .*

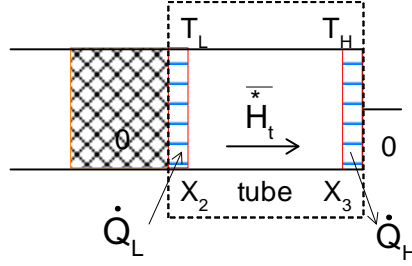


Figure 6: *System containing the tube and the two adjacent heat exchangers.*

temperature of the gas at a certain point in the regenerator is constant in time. As a result of Eqs.(6) and (4) the average enthalpy flow in the regenerator is zero

$$\overline{H_r^*} = 0. \quad (8)$$

The enthalpy flow at the hot end of the system is zero as well. The first law then gives

$$\overline{Q_L} = -\overline{Q_H}. \quad (9)$$

This important relation shows that the cooling power is equal to the heat released at the warm heat exchanger  $X_3$ .

Now consider a system consisting of the regenerator and its two adjacent heat exchangers  $X_1$  and  $X_2$  as in Fig.7. The second law, applied to  $X_2$ , gives

$$\overline{Q_L} = -T_L \overline{S_r^*}, \quad (10)$$

where  $\overline{S_r^*}$  is the entropy flow in the regenerator. It is counted positive when going from left to right. The negative sign means that the average entropy flow in the regenerator is directed from the *cold* to the

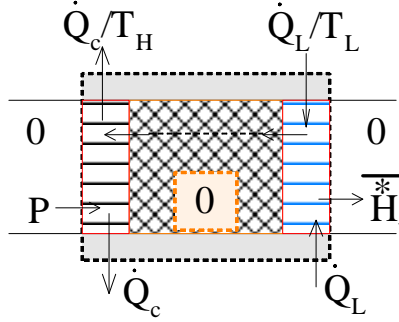


Figure 7: Entropy and enthalpy flows in an ideal regenerator.

warm end. This is logical since the gas moves to the right when the pressure is high (low  $S_m$ , see Eq.(41)) and to the left when the pressure is low ( $S_m$  high). In the ideal case there is no entropy production in the regenerator and  $\bar{S}_r^*$  is constant. In  $X_1$  heat  $\bar{Q}_c$  is removed. The gas in the adiabatic compressor moves back and forth isentropically, so the average entropy flow at the compressor side is zero. The second law for  $X_1$  prescribes

$$\bar{Q}_c = -T_H \bar{S}_r^*. \quad (11)$$

At the compressor side enthalpy flows from the piston to  $X_1$ . The first law for the system containing the compressor and  $X_1$  demands

$$\bar{Q}_c = P. \quad (12)$$

Here  $P$  is the power input in the compressor. The Coefficient Of Performance ( $COP$ ) is defined as

$$\xi = \frac{\bar{Q}_L}{P}. \quad (13)$$

Combining Eqs.(10), (11), and (12) gives the well-known result

$$\xi = \frac{T_L}{T_H}. \quad (14)$$

Comparing this value with the Carnot  $COP$

$$\xi_C = \frac{T_L}{T_H - T_L} \quad (15)$$

shows that the efficiency of an ideal PTR is less than of an ideal cooler. This is due to the dissipation in the orifice. Close to room temperature this difference between  $\xi$  and  $\xi_C$  is big, but at temperatures in the range of liquid nitrogen (78 K) and lower the difference in  $COP$  is insignificant.

### 3 Joule-Thomson coolers

The Joule-Thomson (JT) cooler is invented by Carl von Linde and William Hampson and is sometimes named after them. Basically it is a very simple type of cooler which is widely applied as the (final stage) of liquefaction machines. It can easily be miniaturized, but it is also used on a very large scale in the



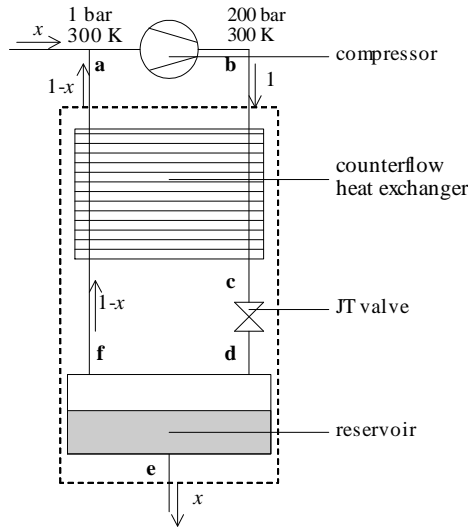


Figure 8: *Schematic diagram of a JT liquefier. At the liquid side a fraction  $x$  of the compressed gas is removed as liquid. At room temperature it is supplied, so that the system is in the steady state. The symbols  $a..f$  refer to points in the  $Ts$ -diagram.*

liquefaction of natural gas. A schematic diagram of a JT liquefier, is given in Fig.8. Basically it consists of a compressor, a counterflow heat exchanger, a JT valve, and a reservoir. In this discussion we assume that the heat exchanger is ideal. This means that it has no flow resistance and that the gas at the low-pressure side leaves the heat exchanger with a temperature equal to room temperature. In Fig.8 the pressures and temperatures refer to the case of a nitrogen liquefier. At the inlet of the compressor the gas is at room temperature (300 K) and a pressure of 1 bar (point a). After compression it is at 300 K and a pressure of 200 bar (point b). Next it enters the heat exchanger where it is pre-cooled. It leaves the exchanger at point c. After the JT expansion, at point d, it has a temperature of 78 K and a pressure of 1 bar. The liquid fraction is  $x$ . The liquid leaves the system at the bottom of the reservoir (point e) and the gas fraction flows into the counterflow heat exchanger at the cold side (point f). As said before it leaves the heat exchanger with a temperature which is equal to room temperature (point a). In the  $Ts$ -diagram of nitrogen the corresponding points are indicated (Fig.9) .

In the steady state a fraction  $x$  of the nitrogen is liquefied after the JT expansion and leaves the system at point e. The same amount is supplied to the system at room temperature. Now we will calculate  $x$ . For this we need the thermodynamic properties of nitrogen which can be obtained from Fig.10. It is convenient to put the relevant data in a table as in Table I. At this point of the derivation we don't know yet the temperature at the high-pressure exit of the heat exchanger (point c). Neither do we know the value of  $x$ , so we don't know the enthalpy and entropy at the points c and d. We can only calculate them after we have determined  $x$ . In order to indicate this the values have been put in between brackets.

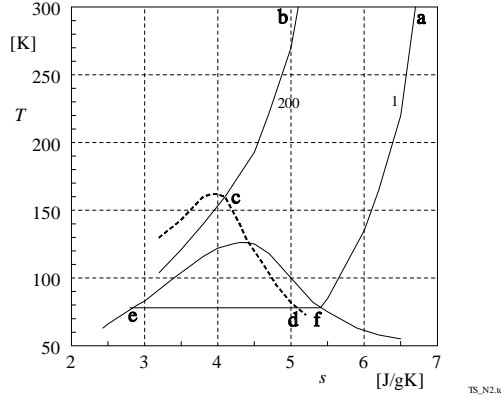


Figure 9:  $Ts$ -diagram of nitrogen with isobars at 1 and 200 bar, the coexistence line and the isenthalp of the JT-expansion indicated. The various points correspond with the points in Fig.8.

	$p$ (bar)	$T$ (K)	$h$ (J/g)	$s$ (J/gK)
a	1	300	555	6.7
b	200	300	525	5.1
c	200	(165)	(307)	(5.2)
d	1	78	(307)	(4.2)
e	1	78	130	2.8
f	1	78	320	5.4

Table I Values of the relevant thermodynamic parameters in the JT cooler. The values in brackets have been obtained after calculation of  $x$ .

The calculation of  $x$  becomes surprisingly simple if we take the proper system. In this case we consider the system indicated by the dotted rectangle in Fig.8. This is an adiabatic system (no heat exchange with its surroundings) with rigid walls and in the steady state. For such a system the first law reduces to a conservation law for the enthalpy. This reads as follows

$$h_b = xh_e + (1 - x)h_a \quad (16)$$

or

$$x = \frac{h_a - h_b}{h_a - h_e}. \quad (17)$$

Clearly there can only be liquefaction if  $x > 0$ . As  $h_a > h_e$  this means

$$h_a > h_b. \quad (18)$$

This is true for nitrogen at room temperature. For every substance there is a certain temperature  $T_i$ , the so-called *inversion temperature*, where  $(\partial h / \partial p)_T$  changes sign. For a starting temperature below the inversion temperature liquefaction is possible via a simple JT expansions.

With the enthalpy values of Table I we get from Eq.(17)

$$x = \frac{555 - 525}{555 - 130} = 0.07. \quad (19)$$

Now the value of the enthalpy in point d can be calculated with

$$h_d = xh_e + (1 - x)h_f \quad (20)$$

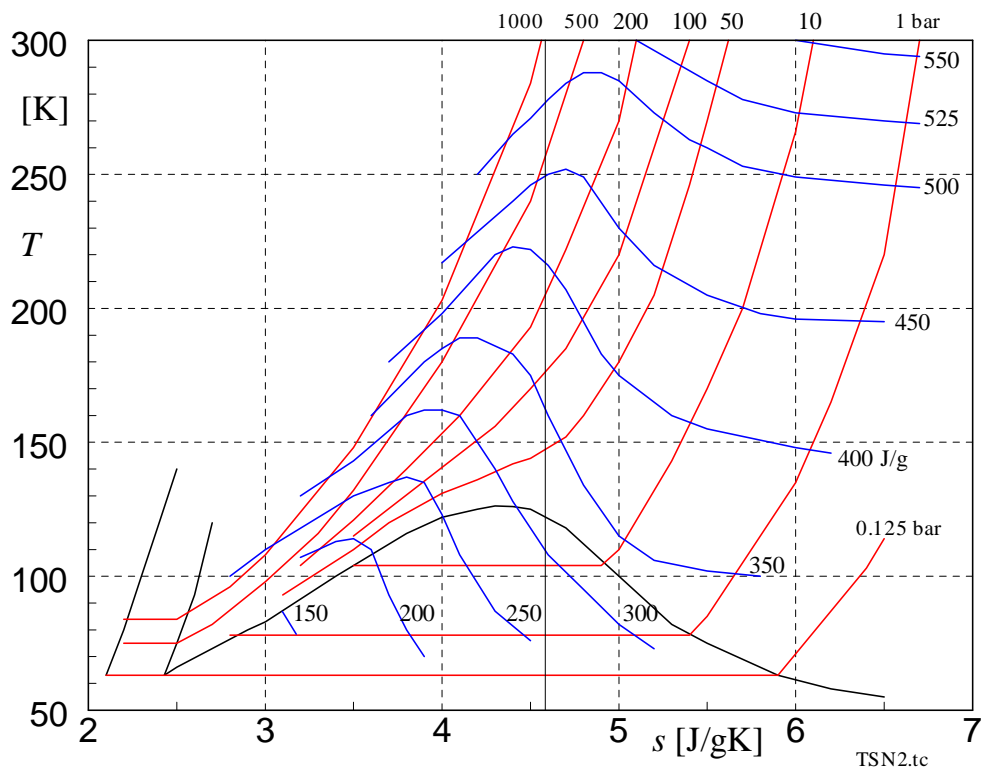


Figure 10:  $Ts$ -diagram of nitrogen with isobars, isenthalps, and the lines of coexistence. The pressures are given in bar, the specific enthalpy in J/g.

or

$$h_d = 307 \text{ J/g.} \quad (21)$$

Similarly

$$s_d = x s_e + (1 - x) s_f \quad (22)$$

or

$$s_d = 5.2 \text{ J/gK.} \quad (23)$$

The enthalpy of point d is equal to the enthalpy in point c. Following the isenthalp of 307 J/g (which runs "parallel" with the isenthalp of 300 J/g) we find the starting temperature of the expansion at point c. It is about  $T_c = 165 \text{ K}$ . The corresponding entropy can be read from the diagram  $s_c = 4.2 \text{ J/gK}$ .

There are several irreversible processes in this cooler. The entropy produced at the expansion is

$$s_{\text{dex}} = s_d - s_c = 5.2 - 4.2 = 1.0 \text{ J/gK.} \quad (24)$$

The total entropy production is

$$s_{\text{itot}} = x s_e + (1 - x) s_a - s_b \quad (25)$$

or

$$s_{\text{itot}} = 0.07 \times 2.8 + (1 - 0.07) \times 6.7 - 5.1 = 1.33 \text{ J/gK.} \quad (26)$$

The difference between  $s_{\text{itot}}$  and  $s_{\text{dex}}$  is due to the irreversible processes in the heat exchanger. To understand the source of the entropy production we must focus on the low-temperature side of the heat exchanger. There the high-pressure gas leaves the heat exchanger with temperature  $T_c = 165 \text{ K}$  while the temperature of the gas entering the heat exchanger is  $T_e = 78 \text{ K}$ . This difference in temperatures between the two sides is fundamental and cannot be avoided even in the case of an ideal heat exchanger. The heat flow over a finite temperature difference is an irreversible process and produces entropy.

If the JT-cooler is used as a cooler (and not as a liquefier) irreversible processes can be suppressed by using gas mixtures instead of pure fluid as pure nitrogen. Also the high pressure values, which are 200 bar in the case of pure nitrogen, side can be reduced significantly.

## 4 Stirling Coolers

### 4.1 Fundamental cycle

The ideal Stirling cooler is depicted schematically in Fig.11. From left to right it consists of:

1. A *warm piston* at room temperature. Here the compression of the gas takes place. The thermal contact with the surroundings at temperature  $T_H$  is supposed to be so good that the compression is isothermal.
2. A *regenerator*.
3. A *cold piston* in the low-temperature region. Here the expansion of the gas takes place. The work, performed during this expansion, is recovered. The thermal contact with the application at temperature  $T_L$  is so good that the expansion is isothermal.

The cooling process can be divided in four steps as depicted in Fig.12. We start the cycle when the two pistons are in their most left position; the cold piston is in contact with the regenerator:

1. From a to b. The warm piston moves to the right over a certain distance while the position of the cold piston is fixed. The pressure  $p$  increases. The compression at the hot end is isothermal so a certain amount of heat  $Q_H$  is released to the surroundings.

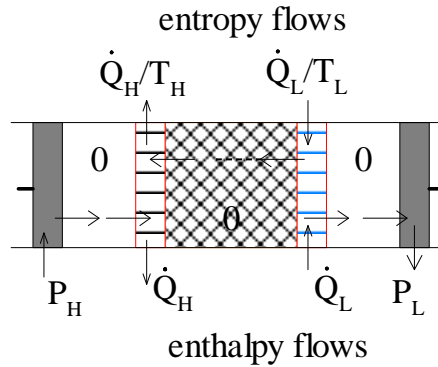


Figure 11: *Schematic diagram of a Stirling cooler. The system has one piston at room temperature and one piston at low temperature. Also shown are the entropy and enthalpy flows.*

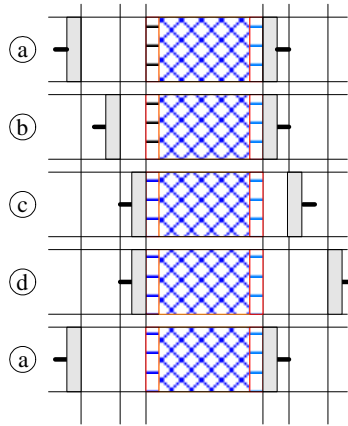


Figure 12: *Four states in the Stirling cycle.*

2. From b to c. Both pistons move to the right so that the total volume remains constant. The gas enters the regenerator at the left with temperature  $T_H$  and leaves it at the right with temperature  $T_L$ . During this part of the cycle  $p$  drops and heat is given off by the gas to the regenerator material.
3. From c to d. The cold piston moves to the right while the position of the warm piston is fixed. The pressure  $p$  is reduced further. The expansion in the cold space is isothermal so heat  $Q_L$  is taken up from the application.
4. From d to a. Both pistons move to the left so that the total volume remains constant. The gas enters the regenerator at the right with temperature  $T_L$  and leaves it at the left with room temperature  $T_H$  so heat is taken up from the regenerator material. The pressure increases. In the end of this step the state is the same as at the start.

In the  $pV$  diagram (Fig.13) the cycle consists of the well-known form of two isotherms and two isochores. The volume  $V$  in this diagram is the total volume between the two pistons. At each point of

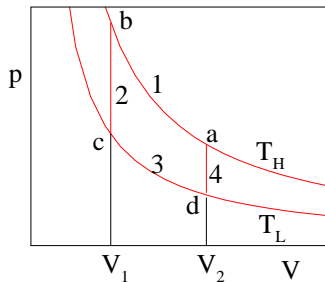


Figure 13:  $pV$ -diagram of the ideal Stirling cycle.

the cycle the pressure in the system and the volume are well-defined. However, during the steps 2 and 4 the temperature of the gas is not homogeneous. Part of the gas is at  $T_H$  and the other part is at  $T_L$ .

In practice the cycle is not divided in discrete steps as described above. Usually the motion of both pistons are driven by a common rotary axes which makes the motions of the two pistons harmonic. It is typical for the Stirling cycle that the phase difference between the motion of the two pistons is about  $90^\circ$ .

The cycle is reversible and heat is exchanged, *with the surroundings*, only at two fixed temperatures, so the efficiency is the Carnot efficiency given by Eq.(15). This can also be seen as follows. In Fig.11 are given the entropy and enthalpy flows. Based on the first law we have for the power supplied to the warm piston

$$P_H = \dot{Q}_H \quad (27)$$

and for the power recovered at the cold piston

$$P_L = \dot{Q}_L \quad (28)$$

and second law

$$\frac{\dot{Q}_H}{T_H} = \frac{\dot{Q}_L}{T_L} \quad (29)$$

resulting in an efficiency

$$\xi = \frac{\dot{Q}_L}{P_H - P_L} = \frac{T_L}{T_H - T_L}. \quad (30)$$

In deriving Eq.(30) we have assumed that the power  $P_L$  released by the cold piston is used to reduce the net power supplied to the system.

## 4.2 Displacer-type

The cold piston as described above is rather impractical. In many cases the cold expander is replaced by a displacer. The principle is shown in Fig.14. A displacer is some metallic volume which moves back and forth with the same frequency as the piston. Typically it is ninety degrees out of phase. Ideally the pressure over the displacer is zero, so no work is required for moving the displacer. Its function is to drive the gas to the warm end of the system (when it is in its most right position) or to the cold end (if it is moved to the left).

We start our description of a cycle in situation (a) where the piston is in its most left position and the displacer at its most right position. All the gas is at room temperature. We split the cooling process in four steps:

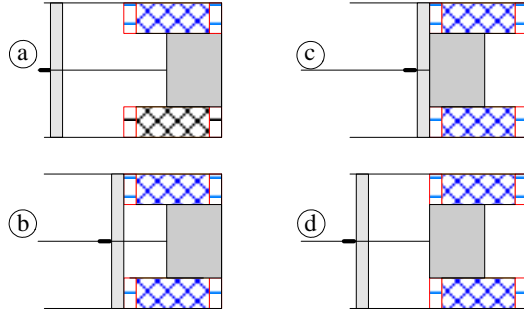


Figure 14: *Modified Stirling cycle. The cold piston in Fig.11 is replaced by a displacer.*

1. From a to b. The piston moves to the right while the position of the displacer is fixed. The compression is isothermal so heat is  $Q_H$  given off to the surroundings.
2. From b to c. The displacer moves to the left while the position of the piston is fixed. The volume behind the piston remains constant. The warm gas is forced to pass the regenerator. It enters the regenerator at temperature  $T_H$  at the left and leaves it at the right with temperature  $T_L$  so heat is given off by the gas to the regenerator material.
3. From c to d. The piston moves to the left with the position of the displacer fixed. The expansion in the cold space is isothermal so heat  $Q_L$  is taken up from the application. However, in contrast with the fundamental Stirling cycle described above, only part of the gas flows through the regenerator to the warm end during this step.
4. From d to a. The displacer moves to the right while the position of the piston is fixed so that the total volume remains constant. The cold gas is forced to pass the regenerator. It enters the regenerator at temperature  $T_L$  and leaves it at  $T_H$  so heat is taken up from the regenerator material. In the end of this step the state is the same as at the start.

In Fig.15 we plotted the pressure against the volume of the system. The cycle consists of one isotherm (step 1) and two isochores (steps 2 and 3). However, during step 4, part of the gas is at a temperature  $T_H$  and the other part at  $T_L$ . The relative amount varies during step 4, so the process line in the pV-diagram is not an isotherm. The cycle is reversible and heat is exchanged, *with the surroundings*, only at two fixed temperatures, so the efficiency of the process is also the Carnot efficiency given by Eq.(15).

### 4.3 Split pair type

Another type of Stirling cooler is the split-pair type as shown in Fig.16. It consists of a compressor, a split tube, and a cold finger. The compressor in Fig.16 is a linear compressor where the piston is driven by an alternating magnetic field comparable to the driving mechanism in loudspeakers. There are two pistons which move in opposite directions to reduce mechanical vibrations. The pistons are suspended by so-called flexure bearings which provide stiffness in the radial direction and flexibility in the axial directions. The piston and the compressor casing don't touch so no lubricants are needed. The slit between the pistons and the casing is very small in order to reduce leakage of gas along the piston.

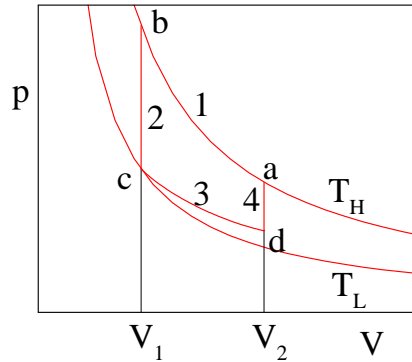


Figure 15:  $pV$ -diagram of the Stirling cycle with displacer. Here  $V$  represents the total volume of the system behind the piston.

In the cold finger the moving part is the regenerator which works as the displacer at the same time. It is suspended by a spring which can also be a flexure bearing. The motion of the displacer/regenerator is driven by the pressure drop between the upper space and the lower space in the cold finger. The cooler operates at a frequency near the resonance frequency of the mass-spring system inside the cold finger.

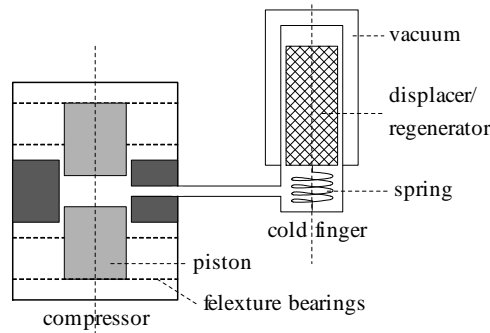


Figure 16: Schematic diagram of a split pair Stirling refrigerator.

## 5 GM-coolers

Fig.17 is a schematic diagram of a Gifford-McMahon cooler. The cold head consists of a regenerator and a displacer just like in the displacer-type Stirling refrigerator. However, the varying pressure is obtained by connecting the system periodically to the high- and low-pressure sides of a compressor. The compression heat is removed by the cooling water of the compressor, in Fig.17 represented by a heat exchanger to room temperature, separate from the compressor.

A cycle can be described as in Fig.18 follows:

The starting position is with the low-pressure valve closed, the high-pressure valve open, and the displacer all the way to the right (so in the cold region). All the gas is at room temperature.



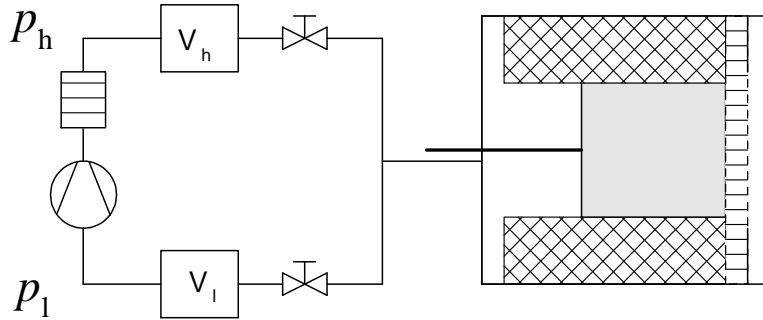


Figure 17: *Schematic diagram of a GM-cooler.  $V_l$  and  $V_h$  are buffer volumes of the compressor. The two valves alternately connect the cooler to the high- and the low-pressure side of the compressor. Usually the two valves are replaced by a rotating valve.*

1. The displacer moves to the left while the high-pressure valve remains open. The gas is forced to pass the regenerator. It enters the regenerator at room temperature  $T_H$  at the left and leaves it at the right with temperature  $T_L$  so heat is given off by the gas to the regenerator material. Due to the high density of the low-temperature gas some additional gas will flow from the high-pressure side of the compressor through the regenerator to the low-temperature volume.
2. The high-pressure valve is closed and the low-pressure valve is opened while the position of the displacer remains fixed. Part of the gas flows through the regenerator to the low-pressure side of the compressor. Heat is taken up from the regenerator material. In the cold space expansion of the gas takes place. The expansion in the cold space is isothermal so heat  $Q_L$  is taken up from the application.
3. The displacer moves to the right while the low-pressure valve remains open. The cold gas is forced to pass the regenerator. It enters the regenerator at temperature  $T_L$  and leaves it at room temperature  $T_H$  so heat is taken up from the regenerator material.
4. The low-pressure valve is closed and the high-pressure valve is opened while the position of the displacer is fixed. In the end of this step the cycle is closed.

During the opening and closing of the valves irreversible processes take place, so GM-coolers are intrinsically irreversible. This is a clear disadvantage of this type of machine. On the other hand the cycle frequencies of the compressor and the displacer are decoupled. E.g. the compressor runs at 50 Hz while the cycle of the displacer and the valves is 1 Hz. This means that the swept volume of the compressor can be 50 times smaller than of the cooler. Basically cheap compressors of home refrigerators can be used, but special precautions have to be taken to prevent overheating of the compressor since they are not designed for helium. In addition very high quality purification traps have to be installed to prevent oil vapor from entering the regenerator.

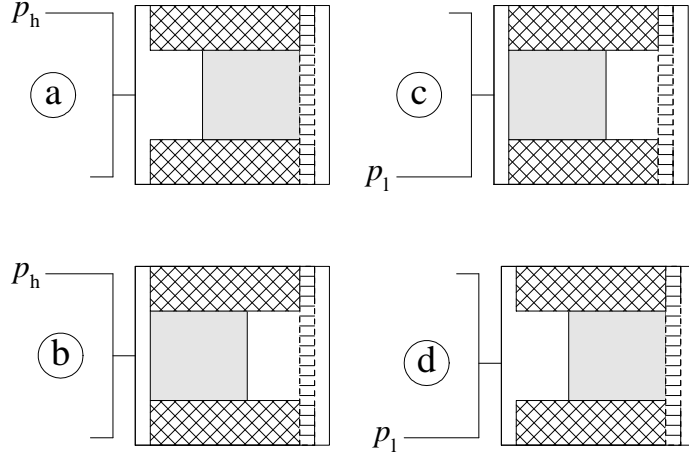


Figure 18: *The four stages in the cooling cycle of the GM cooler.*

## 6 Appendix: Ideal-gas relations

In most of the cases helium is used as the working fluid. At temperatures above about 20 K and at pressures below 30 bar helium more or less behaves as an ideal gas. If the fluid is an ideal gas then

$$pV_m = RT \quad (\text{ideal gas}) \quad (31)$$

giving

$$T\alpha_V = 1 \quad (\text{ideal gas}). \quad (32)$$

The relation for the entropy becomes

$$dS_m = C_p \frac{dT}{T} - R \frac{dp}{p} \quad (\text{ideal gas}). \quad (33)$$

Other useful expression for the entropy are

$$dS_m = C_V \frac{dT}{T} + R \frac{dV_m}{V_m} \quad (\text{ideal gas}) \quad (34)$$

and

$$dS_m = C_p \frac{dV_m}{V_m} + C_V \frac{dp}{p} \quad (\text{ideal gas}). \quad (35)$$

The latter expression immediately results in the well-known Poisson equation for the isentropic expansion of an ideal gas

$$p_2 V_{m2}^\gamma = p_1 V_{m1}^\gamma \quad (\text{ideal gas, } S_m \text{ constant}) \quad (36)$$

with

$$\gamma = \frac{C_p}{C_V}. \quad (37)$$

From Eq.(34)

$$T_2 V_{m2}^{R/C_V} = T_1 V_{m1}^{R/C_V} \quad (\text{ideal gas, } S_m \text{ constant}) \quad (38)$$

and from Eq.(33)

$$T_2 = T_1 \left( \frac{p_2}{p_1} \right)^{R/C_p} \quad (\text{ideal gas, } S_m \text{ constant}). \quad (39)$$

For helium with  $R/C_p = 0.4$

$$T_2 = T_1 \left( \frac{p_2}{p_1} \right)^{0.4} \quad (\text{ideal gas, } S_m \text{ constant, helium}). \quad (40)$$

An analytical expression for the molar entropy of an ideal gas is

$$S_m(p, T) = S_m(p_0, T_0) + C_p \ln \frac{T}{T_0} - R \ln \frac{p}{p_0} \quad (\text{ideal gas}). \quad (41)$$

For the enthalpy we may write

$$H_m(p, T) = C_p T \quad (\text{ideal gas}). \quad (42)$$